A SIMPLE PROOF OF THE MATRIX-VALUED FEJÉR-RIESZ THEOREM

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Abstract. A very short proof of the Fejér-Riesz lemma is presented in the matrix case.

The following fundamental result in matrix spectral factorization theory belongs to Wiener [8] (see also [5], [4]):

Theorem. *Let*

$$S(z) \sim \sum_{n=-\infty}^{\infty} \sigma_n z^n, \tag{1}$$

$|z| = 1$, *σ_k are $r\times r$ matrix coefficients, be a positive definite matrix-function with integrable entries, $S(z) \in L_1(\mathbb{T})$. If the logarithm of the determinant is integrable,* $\log\det S(z) \in L_1(\mathbb{T})$, *then there exists a factorization*

$$S(z) = \chi^+(z)(\chi^+(z))^*, \tag{2}$$

where

$$\chi^+(z) = \sum_{n=0}^{\infty} \rho_n z^n, \tag{3}$$

$|z| < 1$, *is an analytic matrix-function with entries from the Hardy space H_2, $\chi^+(z) \in H_2$, and the determinant of which is an outer function.*

The relation (2) is assumed to hold a.e. on the unit circle $\mathbb{T}$ and $(\chi^+)^* = (\overline{\chi^+})^T$ is the adjoint of χ^+.

It is well-known that if $S(z)$ in (1) is a Laurent polynomial of order m, then $\chi^+(z)$ in (3) is a polynomial of the same order m. This result is known as the Fejér-Riesz lemma in the scalar case and it was generalized by M. Rosenblatt [7] and Helson [4] to the matrix case using the linear prediction theory of multidimensional weakly stationary processes as in the proof of the existence theorem above. One can find a constructive but long proof of the matrix Fejér-Riesz lemma in [3] as well. Below, we present a simple, transparent and natural proof of this assertion.

Theorem 1. *If $S(z) = \sum_{n=-m}^{m} \sigma_n z^n$, then $\chi^+(z) = \sum_{n=0}^{m} \rho_n z^n$.*

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We use the generalization of Smirnov's theorem (see [6], p. 109) which claims that if the boundary values of an analytic function $f(z)=g(z)/h(z)$, where $g\in H_{p_1}$ and h is an outer function from H_{p_2}, belongs to $L_p(\mathbb{T})$, then $f\in H_p$.

Proof. We know that

$$(\chi^+(z))^*\in L_2^-(\mathbb{T}) \tag{4}$$

in general, and it suffices to show that

$$z^m(\chi^+(z))^*\in L_2^+(\mathbb{T})=H_2, \tag{5}$$

where $L_2^-(\mathbb{T})$ and $L_2^+(\mathbb{T})$ are the classes of square integrable functions with, respectively, positive and negative Fourier coefficients equal to 0, and the latter is naturally identified with H_2.

It follows from (2) that

$$(\chi^+(z))^{-1}z^mS(z)=z^m(\chi^+(z))^* \tag{6}$$

for a.a. $z\in\mathbb{T}$. The matrix-function

$$(\chi^+(z))^{-1}=\frac{1}{\det\chi^+(z)}A(z)$$

is analytic in the unit circle, where $A(z)\in H_{2/r}$. Consequently, since $z^mS(z)\in H_\infty$ by hypothesis, the entries of the left-hand side matrix in (6) can be represented as the ratios of some functions from $H_{2/r}$ and the outer function $\det\chi^+(z)\in H_{2/r}$, while their boundary values belong to $L_2(\mathbb{T})$ because of (6) and (4). Thus, by virtue of the above mentioned generalization of Smirnov's theorem, $(\chi^+(z))^{-1}z^mS(z)\in H_2=L_2^+(\mathbb{T})$ and (5) follows again from (6). □

The same idea can be used to prove the uniqueness (up to a constant unitary matrix) of the spectral factorization (2). Indeed, assume $S(z)=\chi_1^+(z)(\chi_1^+(z))^*$ together with (2), where $\chi_1^+(z)\in H_2$ and $\det\chi_1^+(z)$ is outer. Then

$$(\chi^+(z))^{-1}\chi_1^+(z)((\chi^+(z))^{-1}\chi_1^+(z))^*=I, \tag{7}$$

so that the analytic matrix-function $U(z):=(\chi^+(z))^{-1}\chi_1^+(z)$, $|z|<1$, is unitary for a.a. $z\in\mathbb{T}$. Thus, the boundary values of $U(z)$ belongs to L_∞ and, as in the proof of Theorem 1, we have $U(z)\in H_\infty$. By changing the roles of χ^+ and χ_1^+ in this discussion, we get $(\chi_1^+(z))^{-1}\chi^+(z)\in H_\infty$. But $(U(z))^*=(\chi_1^+(z))^{-1}\chi^+(z)$ for a.a. $z\in\mathbb{T}$, by virtue of (7). Thus the boundary values of $U(z)$ as well as its conjugate belongs to $L_\infty^+(\mathbb{T})$ which implies that $U(z)$ is constant.

A further development of the circle of ideas presented in this paper leads to the constructive analytic proof of the existence theorem, formulated in the beginning, as well as to the efficient algorithm for approximate computation

of the spectral factor (3) for a given matrix-function (1) (see [2], [1]). Such an algorithm is very important for practical applications and it has been searched since Wiener's period.

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